

Conformally flat Lorentzian manifolds with special holonomy groups

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Abstract

It is shown that among conformally flat Lorentzian manifolds there are four classes of spaces with special holonomy groups: pp-waves with a certain potential and three classes of spaces with the holonomy group $\text{Sim}(n)$, which can be described as extensions of Riemannian spaces of constant sectional curvature to certain Walker metrics.

1 Introduction and the main results

It is known [14] that a conformally flat Riemannian manifold is either a product of two spaces of constant sectional curvature, or it is a product of a space of constant sectional curvature with an interval, or its restricted holonomy group is the identity component of the orthogonal group. The last condition represents the general case and among various manifolds satisfying the last condition one can emphasize only the spaces of constant sectional curvature.

One usually says that the connected holonomy group of an indecomposable pseudo-Riemannian manifold is special if it is different from the connected component of the pseudo-orthogonal group [4]. In this case the holonomy group defines a special geometry on the manifold. For example, a pseudo-Riemannian manifold of signature (r, s) is pseudo-Kählerian if and only if its holonomy group is contained in $U(\frac{r}{2}, \frac{s}{2})$. We see that there are no conformally flat Riemannian manifolds with special holonomy groups.

In the case of pseudo-Riemannian manifolds, the holonomy group can be weakly irreducible, this means that it does not preserve any non-degenerate proper vector subspace of the tangent space, and not irreducible in the same time, i.e. it may preserve a degenerate vector subspace of the tangent space.

The main result of the present paper is the complete local description of conformally flat Lorentzian manifolds (M, g) with weakly irreducible not irreducible holonomy groups, which are the only special holonomy groups of Lorentzian manifolds. Let $\dim M = n + 2 \geq 4$. The holonomy algebra, i.e. the Lie algebra of the holonomy group, $\mathfrak{g} \subset \mathfrak{so}(1, n + 1)$ of such manifold preserves an isotropic line of the tangent space, which is identified with the Minkowski space $\mathbb{R}^{1, n+1}$. Hence \mathfrak{g} is contained in the maximal subalgebra of $\mathfrak{so}(1, n + 1)$ preserving an isotropic line. This algebra is denoted by $\mathfrak{sim}(n)$ and it admits the decomposition

$$\mathfrak{sim}(n) = (\mathbb{R} \oplus \mathfrak{so}(n)) \ltimes \mathbb{R}^n.$$

Any manifold (M, g) with such holonomy algebra (locally) admits a distribution of isotropic lines. Such manifolds are called the Walker manifolds [3, 18]. On any such manifold (M, g) there exist coordinates v, x^1, \dots, x^n, u and the metric g has the form

$$g = 2dvdu + h + 2Adu + H(du)^2, \quad (1)$$

where $h = h_{ij}(x^1, \dots, x^n, u)dx^i dx^j$ is an u -dependent family of Riemannian metrics, $A = A_i(x^1, \dots, x^n, u)dx^i$ is an u -dependent family of one-forms, and H is a local function on M . The vector field ∂_v defines the parallel distribution of isotropic lines. An important class of Walker manifolds form pp-waves that are given locally by (1) with $A = 0$, $h = \sum_i (dx^i)^2$, and $\partial_v H = 0$, see [15] and references there. The pp-waves are exactly Walker manifolds with the commutative holonomy algebra $\mathfrak{g} \subset \mathbb{R}^n \subset \mathfrak{sim}(n)$.

Theorem 1 *Let (M, g) be a conformally flat Lorentzian manifold of dimension $n + 2 \geq 4$. Then the holonomy algebra \mathfrak{g} of (M, g) is weakly irreducible and not irreducible if and only if one of the following holds:*

- 1) $\mathfrak{g} = \mathbb{R}^n \subset \mathfrak{sim}(n)$, i.e. (M, g) is a pp-wave, and locally there exist coordinates v, x^1, \dots, x^n, u and a function $a(u)$ such that

$$g = 2dvdu + \sum_{i=1}^n (dx^i)^2 + a(u) \sum_{i=1}^n (x^i)^2 (du)^2,$$

and $a(u) \neq 0$ for some system of coordinates;

- 2) $\mathfrak{g} = \mathfrak{sim}(n)$ and locally there exist coordinates v, x^1, \dots, x^n, u and functions $a(u)$, $B_i(u)$, $C_i(u)$, $D(u)$ such that

$$g = 2dvdu + \sum_{i=1}^n (dx^i)^2 + 2Adu + (vH_1 + H_0)(du)^2,$$

where

$$A = A_i dx^i, \quad A_i = \frac{1}{4} \left(2B_j(u) x^j x^i - B_i(u) \sum_{j=1}^n (x^j)^2 \right),$$

$$H_1 = B_j(u) x^j,$$

$$H_0 = \frac{1}{16} \sum_{k=1}^n B_k^2(u) \left(\sum_{i=1}^n (x^i)^2 \right)^2 + a(u) \sum_{i=1}^n (x^i)^2 + C_i(u) x^i + D(u),$$

and $\sum_i B_i^2(u) \neq 0$ for some system of coordinates;

- 3) $\mathfrak{g} = \mathfrak{sim}(n)$ and locally there exist coordinates v, x^1, \dots, x^n, u and functions $\lambda(u) < 0$, $a(u)$, $B_i(u)$, $C_i(u)$, $D(u)$ such that

$$g = 2dvdu + \frac{1}{-\lambda(u)} \Psi \sum_{i=1}^n (dx^i)^2 + 2Adu + \lambda(u)(v^2 - H_0)(du)^2,$$

where

$$\Psi = \frac{4}{(1 + \sum_{k=1}^n (x^k)^2)^2},$$

$$A = A_i dx^i, \quad A_i = \Psi \left(B_j(u) x^j x^i - \frac{1}{2} B_i(u) \sum_{j=1}^n (x^j)^2 + \frac{1}{2} B_i(u) \right),$$

$$H_0 = \Psi \left(\sum_{k=1}^n B_k^2(u) + (B_k x^k)^2 \right) + \sqrt{\Psi} \left(a(u) \sum_{i=1}^n (x^i)^2 + C_i(u) x^i + D(u) \right),$$

and $\sum_i B_i^2(u) + a^2(u) \neq 0$ for some system of coordinates;

- 4) $\mathfrak{g} = \mathfrak{sim}(n)$ and locally there exist coordinates v, x^1, \dots, x^n, u and functions $\lambda(u) > 0$, $a(u)$, $B_i(u)$, $C_i(u)$, $D(u)$ such that

$$g = 2dvdu + \frac{1}{\lambda(u)} \Psi \sum_{i=1}^n (dx^i)^2 + 2Adu + \lambda(u)(v^2 + H_0)(du)^2,$$

where

$$\begin{aligned} \Psi &= \frac{4}{(1 - \sum_{k=1}^n (x^k)^2)^2}, \\ A &= A_i dx^i, \quad A_i = \Psi \left(B_j(u) x^j x^i - \frac{1}{2} B_i(u) \sum_{j=1}^n (x^j)^2 - \frac{1}{2} B_i(u) \right), \\ H_0 &= \Psi \left(\sum_{k=1}^n B_k^2(u) - (B_k x^k)^2 \right) + \sqrt{\Psi} \left(a(u) \sum_{i=1}^n (x^i)^2 + C_i(u) x^i + D(u) \right), \end{aligned}$$

and $\sum_i B_i^2(u) + a^2(u) \neq 0$ for some system of coordinates

Note that in the cases 3) and 4) the Riemannian metrics $\Psi \sum_{k=1}^n (dx^k)^2$ are the metrics of the sphere and the Lobachevskian space, respectively. Moreover, the u -families of 1-forms A are families of Killing 1-forms on these spaces. These Killing 1-forms have the above shape, since we made a choice of the coordinates in the proof. Generally A can be an arbitrary u -family of Killing 1-forms, but then the function H_0 will be more complicated.

We may describe also all conformally flat Walker metrics, i.e. without the decomposability assumption.

Theorem 2 *Let g be the Walker metric (1). Then g is conformally flat if and only if one of the following holds*

- 1) g is indecomposable and after a coordinate transformation it coincides with one of the metrics of Theorem 1;
- 2) g is decomposable and after a coordinate transformation it coincides with one of the following metrics:

$$\begin{aligned} g &= \frac{1}{c} \Psi \sum_{k=1}^n (dx^k)^2 + 2dvdu - cv^2(du)^2, \quad \Psi = \frac{4}{(1 + \sum_{k=1}^n (x^k)^2)^2}, \quad c > 0 \\ g &= \frac{1}{c} \Psi \sum_{k=1}^n (dx^k)^2 + 2dvdu + cv^2(du)^2, \quad \Psi = \frac{4}{(1 - \sum_{k=1}^n (x^k)^2)^2}, \quad c > 0 \\ g &= \sum_{k=1}^n (dx^k)^2 + 2dvdu. \end{aligned}$$

In Section 2, the result of Kurita [14] is extended to the case of pseudo-Riemannian manifolds. In Section 3, an expression for the Weyl conformal curvature tensor for a Walker metric is given. Section 4 is dedicated to the proof of Theorem 1. In Section 5, the Ricci operator for the metrics from Theorem 1 is computed.

In Section 6, the case of dimension 4 is considered. Possible holonomy algebras of conformally flat 4-dimensional Lorentzian manifolds are classified also in [13]. The first metric from Theorem

1 in dimension 4 is given in [17]. In [13], it is posed the problem to construct an example of conformally flat metric with the holonomy algebra $\mathfrak{sim}(2)$ (which is denoted in [13] by R_{14}). An attempt to construct such metric can be found in [11]. We show that the metric constructed there is in fact decomposable and its holonomy algebra is $\mathfrak{so}(1, 1) \oplus \mathfrak{so}(2)$. Thus in this paper we get metrics with the holonomy algebra $\mathfrak{sim}(n)$ for the first time, and even more, we find all such metrics.

An important fact is that a simply connected conformally flat spin Lorentzian manifold admits the spaces of conformal Killing spinors of maximal dimension [2].

The results of this paper are used in [1] for the classification of Lorentzian manifolds satisfying the condition $\nabla^2 R = 0$.

2 Decomposability of conformally flat pseudo-Riemannian manifolds

In [14], Kurita proved the following theorem for the case of Riemannian manifolds.

Theorem 3 *Let (M, g) be an n -dimensional conformally flat Riemannian manifold. Then its local restricted holonomy group H_x ($x \in M$) is in general $\mathrm{SO}(n)$. If $H_x \neq \mathrm{SO}(n)$, then for some coordinate neighborhood U of x one of the following holds:*

- 1) H_x is identity and the metric is flat in U ;
- 2) $H_x = \mathrm{SO}(k) \times \mathrm{SO}(n-k)$ and U is a direct product of a k -dimensional manifold of constant sectional curvature K and an $(n-k)$ -dimensional manifold of constant sectional curvature $-K$ ($K \neq 0$);
- 3) $H_x = \mathrm{SO}(n-1)$ and U is a direct product of a straight line (or a segment) and an $(n-1)$ -dimensional manifold of constant sectional curvature.

We generalize this theorem for the case of pseudo-Riemannian manifolds. We also make it more precise.

Theorem 4 *Let (M, g) be a conformally flat pseudo-Riemannian manifold of signature (r, s) with the restricted holonomy group $\mathrm{Hol}^0(M, g)$. If (M, g) is not flat, then one of the following holds:*

- 1) $\mathrm{Hol}^0(M, g) = \mathrm{SO}(r, s)$;
- 2) $\mathrm{Hol}^0(M, g)$ is weakly irreducible and not irreducible (in particular, it preserves a degenerate subspace of the tangent space);
- 3) $\mathrm{Hol}^0(M, g) = \mathrm{SO}(r_1, s_1) \times \mathrm{SO}(r - r_1, s - s_1)$ and each point $x \in M$ has a neighborhood that is either flat or it is a product of a pseudo-Riemannian manifold of constant sectional curvature K and signature (r_1, s_1) and a pseudo-Riemannian manifold of constant sectional curvature $-K$ ($K \neq 0$) and signature $(r - r_1, s - s_1)$;
- 4) $\mathrm{Hol}^0(M, g) = \mathrm{SO}(r - 1, s)$ (resp., $H_x = \mathrm{SO}(r, s - 1)$) and each point $x \in M$ has a neighborhood that is either flat or it is a product of a pseudo-Riemannian manifold of constant sectional curvature and signature $(r - 1, s)$ (resp., $(r, s - 1)$) and the space $(L, -(dt)^2)$ (resp., $(L, (dt)^2)$), L is the straight line or a segment.

Proof. Let (M, g) be a pseudo-Riemannian manifold of signature (r, s) and dimension $d = r + s$. The vector bundle $\mathfrak{so}(TM)$ of skew-symmetric endomorphisms of the tangent bundle TM can be identified with the space of bivectors $\wedge^2 TM$ in such a way that

$$(X \wedge Y)Z = g(X, Z)Y - g(Y, Z)X$$

for all vector fields X, Y, Z on M . The Weyl tensor W of the pseudo-Riemannian manifold (M, g) is defined by the equality

$$W = R + R_L, \quad (2)$$

where the tensor R_L is defined by

$$R_L(X, Y) = LX \wedge Y + X \wedge LY, \quad (3)$$

$$L = \frac{1}{d-2} \left(\text{Ric} - \frac{s}{2(d-1)} \text{id} \right)$$

is the Schouten tensor and s is the scalar curvature.

Suppose that the restricted holonomy group $\text{Hol}^0(M, g)$ is not weakly irreducible. The Wu decomposition Theorem [19] states that each point of M has a neighborhood U such that $(U, g|_U)$ is a product

$$(U, g|_U) = (M_1 \times M_2, g_1 + g_2)$$

of two pseudo-Riemannian manifolds (M_1, g_1) and (M_2, g_2) . Let d_1 and d_2 be the dimensions of these manifolds. For the curvature tensors, Ricci operators and the scalar curvatures it holds

$$R = R_1 + R_2, \quad \text{Ric} = \text{Ric}_1 + \text{Ric}_2, \quad s = s_1 + s_2.$$

First suppose that $d \geq 4$. In this case $W = 0$ and we get

$$R_1 + R_2 = -R_L. \quad (4)$$

Assume that $d_1 \geq d_2$ and $d_1 \geq 2$. The curvature tensor R_1 can be written in the form $R_1 = W_1 - R_{L_1}$. Considering (4) restricted to TM_1 , we get that $W_1 = 0$ and

$$\frac{1}{d_1 - 2} \left(\text{Ric}_1 - \frac{s_1}{2(d_1 - 1)} \text{id} \right) = \frac{1}{d - 2} \left(\text{Ric}_1 - \frac{s_1 + s_2}{2(d - 1)} \text{id} \right). \quad (5)$$

If $d_2 \geq 2$, then taking the trace in (5), we get

$$\frac{s_1}{d_1(d_1 - 1)} = -\frac{s_2}{d_2(d_2 - 1)}.$$

Since s_1 is a function on M_1 and s_2 is a function on M_2 , the both functions must be constant. Substituting the last equality back to (5), we obtain

$$\text{Ric}_1 = \frac{s_1}{d_1} \text{id}. \quad (6)$$

Next,

$$R_1(X, Y) = \frac{s_1}{d_1(d_1 - 1)} X \wedge Y. \quad (7)$$

The same holds for the second manifold. For the sectional curvatures we get

$$k_1 = \frac{s_1}{d_1(d_1 - 1)} = -\frac{s_2}{d_2(d_2 - 1)} = -k_2.$$

If $d_2 = 1$, than (5) is equivalent to (6) and this implies (7). From this and the Schur Theorem it follows that k_1 is constant. If $d_1 = 2$, then the curvature tensor R_1 satisfies $R_1(X, Y) = fX \wedge Y$ for some function f on M_1 . The proof in this case is the same.

If $d = 3$, then $d_1 = 2$ and $d_2 = 1$. It holds $R = R_1$ and $R_1(X, Y) = fX \wedge Y$ for some function f on M_1 . In this case (M, g) is conformally flat if and only if the Cotton tensor C defined by

$$C(X, Y, Z) = g((\nabla_Z L)X, Y) - g((\nabla_Y L)X, Z)$$

is zero. This implies that f is constant, i.e. (M_1, g_1) has constant sectional curvature.

Now we have to prove that if $\text{Hol}^0(M, g)$ is irreducible, then it coincides with $\text{SO}(r, s)$. Suppose that $\text{Hol}^0(M, g)$ is irreducible and it is different from $\text{SO}(r, s)$ and $\text{U}(\frac{r}{2}, \frac{s}{2})$. Then the manifold is Einstein [4]. Since (M, g) is in addition conformally flat, (M, g) has constant sectional curvature and its connected holonomy group must be either trivial or $\text{SO}(r, s)$, i.e. we get a contradiction. It is known that if a pseudo-Kählerian manifold is conformally flat, then it is flat [20], hence $\text{Hol}^0(M, g) \neq \text{U}(\frac{r}{2}, \frac{s}{2})$. This proves Theorem 4. \square

3 The Weyl curvature tensor of Walker metrics

In order to prove Theorem 1, we give some information about the curvature tensor of the Walker metric (1). For the fixed coordinates v, x^1, \dots, x^n, u consider the fields of frames

$$p = \partial_v, \quad X_i = \partial_i - A_i \partial_v, \quad q = \partial_u - \frac{1}{2} H \partial_v.$$

Consider the distribution $E = \text{span}\{X_1, \dots, X_n\}$. The fibers of this distribution can be identified with the tangent spaces to the Riemannian manifolds with the Riemannian metrics $h(u)$. Denote by R_0 the tensor corresponding to the family of the curvature tensors of $h(u)$ under this identification. Similarly denote by $\text{Ric}(h)$ the corresponding Ricci endomorphism acting on sections of E .

From the results of [9] it follows that the curvature tensor R of the metric g can be written in the form

$$R(p, q) = -\lambda p \wedge q - p \wedge \vec{v}, \quad R(X, Y) = R_0(X, Y) - p \wedge (P(Y)X - P(X)Y), \quad (8)$$

$$R(X, q) = -g(\vec{v}, X)p \wedge q + P(X) - p \wedge T(X), \quad R(p, X) = 0 \quad (9)$$

for all $X, Y \in \Gamma(E)$. Here λ is a function, $\vec{v} \in \Gamma(E)$, $T \in \Gamma(\text{End}(E))$ is symmetric, $T^* = T$, and the tensor $P \in \Gamma(E^* \otimes \mathfrak{so}(E))$ satisfies

$$g(P(X)Y, Z) + g(P(Y)Z, X) + g(P(Z)X, Y) = 0 \text{ for all } X, Y, Z \in \Gamma(E).$$

These element may be found in terms of the coefficients of the metric (1). For example,

$$\lambda = \frac{1}{2} \partial_v^2 H, \quad \vec{v} = \frac{1}{2} (\partial_i \partial_v H - A_i \partial_v^2 H) h^{ij} X_j. \quad (10)$$

Let $P(X_k)X_j = P_{jk}^i X_i$ and $T(X_j) = \sum_i T_{ij} X_j$. Then

$$h_{il} P_{jk}^l = g(R(X_k, q)X_j, X_i), \quad T_{ij} = -g(R(X_i, q)q, X_j).$$

Suppose that h does not depend on u . Using direct computations, we obtain

$$h_{il} P_{jk}^l = -\frac{1}{2} \nabla_k F_{ij}, \quad (11)$$

$$\begin{aligned} T_{ij} = & -\frac{1}{2} \nabla_i \nabla_j H + \frac{1}{4} F_{ik} F_{jl} h^{kl} + \frac{1}{4} (\partial_v H) (\nabla_i A_j + \nabla_j A_i) \\ & + \frac{1}{2} (A_i \partial_j \partial_v H + A_j \partial_i \partial_v H) + \frac{1}{2} \partial_u (\nabla_i A_j + \nabla_j A_i) - \frac{1}{2} A_i A_j \partial_v^2 H, \end{aligned} \quad (12)$$

where

$$F_{ij} = \partial_i A_j - \partial_j A_i$$

is the differential of the 1-form A , and the covariant derivatives are taken with respect to the metric h . In the case of h , A and H independent of u , the curvature tensor of the metric (1) is found in [12].

The Ricci operator has the following form:

$$\text{Ric}(p) = \lambda p, \quad \text{Ric}(X) = -g(X, \widetilde{\text{Ric}} P - \vec{v})p + \text{Ric}(h)(X), \quad (13)$$

$$\text{Ric}(q) = -(\text{tr } T)p - \widetilde{\text{Ric}}(P) + \vec{v} + \lambda q, \quad (14)$$

where $\widetilde{\text{Ric}} P = h^{ij} P(X_i) X_j$ [10]. For the scalar curvature we get $s = 2\lambda + s_0$, where s_0 is the scalar curvature of h . Using this, we may compute the tensor R_L ,

$$R_L(p, X) = \frac{1}{n} p \wedge \left(\text{Ric}(h) + \frac{(n-1)\lambda - s_0}{n+1} \text{id} \right) X, \quad (15)$$

$$R_L(p, q) = \frac{1}{n} \left(\frac{2n\lambda - s_0}{n+1} p \wedge q + p \wedge (\vec{v} - \widetilde{\text{Ric}} P) \right), \quad (16)$$

$$\begin{aligned} R_L(X, Y) = & \frac{1}{n} \left(p \wedge (g(X, \vec{v} - \widetilde{\text{Ric}} P) Y - g(Y, \vec{v} - \widetilde{\text{Ric}} P) X) \right. \\ & \left. + \left(\text{Ric}(h) - \frac{s}{2(n+1)} \right) X \wedge Y + X \wedge \left(\text{Ric}(h) - \frac{s}{2(n+1)} \right) Y \right), \end{aligned} \quad (17)$$

$$\begin{aligned} R_L(X, q) = & \frac{1}{n} \left((\text{tr } T)p \wedge X + g(X, \vec{v} - \widetilde{\text{Ric}} P)p \wedge q + X \wedge (\vec{v} - \widetilde{\text{Ric}} P) \right. \\ & \left. + \left(\text{Ric}(h) + \frac{(n-1)\lambda - s_0}{n+1} \text{id} \right) X \wedge q \right). \end{aligned} \quad (18)$$

The Weyl tensor W can be computed using this and (2).

4 Proof of Theorem 1

Suppose that the holonomy algebra of the Lorentzian manifold (M, g) is weakly irreducible and not irreducible, i.e. it is contained in $\mathfrak{sim}(n)$. Then the local form of the metric g is given by (1). Suppose that g is conformally flat, i.e. $W = 0$.

Lemma 1 *The equation $W = 0$ is equivalent to the following system of equations:*

$$s_0 = -n(n-1)\lambda, \quad R_0 = -\frac{1}{2}\lambda R_{\text{id}}, \quad P(X) = \vec{v} \wedge X, \quad T = f \text{id}_E, \quad (19)$$

where X is any section of E and f is a function. In particular, $W = 0$ implies that $\widetilde{\text{Ric}} P = -(n-1)\vec{v}$ and the Weyl tensor W_0 of h is zero.

Proof. Suppose that $W = 0$. Then it holds $R = -R_L$. From (9) and (15) it follows that $\text{Ric}(h) = -\frac{(n-1)\lambda - s_0}{n+1} \text{id}$. Taking the trace, we get $s_0 = -n(n-1)\lambda$. Hence, $\text{Ric}(h) = \frac{s_0}{n} \text{id}$, i.e. the metrics h are Einstein, and it holds

$$R_0 = W_0 - \frac{s_0}{2n(n-1)} R_{\text{id}}.$$

From (8) and (17) it follows that

$$R_0 + \frac{s_0}{2n(n-1)}R_{\text{id}} = 0.$$

We conclude that $W_0 = 0$. Using (9) and (18), we get $P(X) = -\frac{1}{n}X \wedge (\vec{v} - \widetilde{\text{Ric}} P)$. Applying $\widetilde{\text{Ric}}$, we obtain $\widetilde{\text{Ric}} P = -(n-1)\vec{v}$. Consequently, $P(X) = \vec{v} \wedge X$. From (9) and (18) it follows that $T(X) = \frac{1}{n}(\text{tr } T)X$, i.e. $T = f \text{ id}$ for a function f . Conversely, (19) implies $W = 0$. \square

We will consider several cases.

Case 1. Suppose that $s_0 = 0$. Then $R_0 = 0$, and each metric in the family h is flat. Consequently, changing the coordinates, we may assume that

$$h = \delta_{ij} dx^i dx^j.$$

From (10) and the equality $\lambda = 0$ it follows that $\partial_v^2 H = 0$, hence

$$H = vH_1 + H_0, \quad \partial_v H_1 = \partial_v H_0 = 0.$$

Using (10), we get

$$\vec{v} = \frac{1}{2} \partial_i H_1 \delta^{ij} X_j. \quad (20)$$

Case 1.1. Suppose that $\vec{v} = 0$ for any coordinate system. Then the curvature tensor satisfies $R(E^\perp, E^\perp) = 0$. Consequently, (M, g) is a pp-wave (see e.g. [8]), then g can be written in the form

$$g = 2dvdu + \sum_{i=1}^n (dx^i)^2 + H(du)^2, \quad \partial_v H = 0$$

i.e. $A = 0$, $H_1 = 0$, and $H_0 = H$. We obtain the equation

$$f \delta_{ij} = -\frac{1}{2} \partial_i \partial_j H_0.$$

Taking the trace, we get $f = -\frac{1}{2n} \Delta H_0$, where $\Delta = \sum_i \partial_i^2$ is the Euclidean Laplacian, and we obtain

$$\frac{1}{n} \Delta H_0 \delta_{ij} = \partial_i \partial_j H_0. \quad (21)$$

The general solution of this equation has the form

$$H_0 = a(u) \sum_{i=1}^n (x^i)^2 + C_i(u) x^i + D(u).$$

Consider the new coordinates

$$\tilde{v} = v - \sum_j \frac{db^j(u)}{du} x^j + d(u), \quad \tilde{x}^i = x^i + b^i(u), \quad \tilde{u} = u.$$

We obtain the metric of the same form with

$$\tilde{H}_0 = a(u) \sum_{i=1}^n (x^i)^2 + \tilde{C}_i(u) x^i + \tilde{D}(u),$$

where

$$\tilde{C}_j = -2 \frac{d^2 b^j}{(du)^2} + 2ab^j + C_j, \quad (22)$$

$$\tilde{D} = 2 \frac{dd(u)}{du} + \sum_j \left(\frac{db^j}{du} \right)^2 + a \sum_j (b^j)^2 + C_i b^i + D. \quad (23)$$

Equation (22) implies the existence of $b^j(u)$ such that $\tilde{C}_k = 0$. Using the last equation, we can chose $d(u)$ such that $\tilde{D} = 0$. Thus we may assume that

$$H_0 = a(u) \sum_{i=1}^n (x^i)^2.$$

Case 1.2. Suppose that $\vec{v} \neq 0$ for some coordinate system. Since $P(X) = \vec{v} \wedge X$, we get

$$P_{jk}^i = \delta_{ki} \vec{v}_j - \delta_{kj} \vec{v}_i,$$

where $\vec{v} = \sum_j \vec{v}_j X_j$. Using (20) and (11), we obtain the system of equations

$$\partial_k F_{ij} = -\delta_{ki} \partial_j H_1 + \delta_{kj} \partial_i H_1. \quad (24)$$

These equations can be rewritten in the form

$$\partial_i (\partial_k A_j - \delta_{kj} H_1) - \partial_j (\partial_k A_i - \delta_{ki} H_1) = 0.$$

This system of equation is equivalent to

$$dG^k = 0,$$

where we define the 1-forms

$$G^k = G_i^k dx^i, \quad G_i^k = \partial_k A_i - \delta_{ki} H_1.$$

We conclude that there exist functions f^k such that

$$G_i^k = \partial_i f^k.$$

System (24) takes the form

$$\partial_k A_i - \delta_{ki} H_1 = \partial_i f^k.$$

This implies

$$F = dA = -df, \quad \text{where} \quad f = \sum_k f^k dx^k$$

and

$$A = -f + d\varphi$$

for some function φ . Since $\partial_v^2 H = 0$, the gauge transformation

$$v \mapsto v - \phi$$

changes the metric in the following way [12]:

$$A \mapsto A + d\phi, \quad H_1 \mapsto H_1, \quad H_0 \mapsto H_0 + H_1 \phi + 2\partial_u \phi. \quad (25)$$

Hence, changing the coordinates, we get $dA = -df$. Equations (24) take the form

$$\partial_i A_j + \partial_j A_i = \delta_{ij} H_1. \quad (26)$$

Conversely, this system of equations implies (24).

Consider now (12). Since $T_{ij} = f\delta_{ij}$ for some function f , we get that $f = vf_1 + f_0$, where $\partial_v f_1 = \partial_v f_0 = 0$. Applying ∂_v to (12), we get

$$f_1 \delta_{ij} = -\frac{1}{2} \partial_i \partial_j H_1.$$

As above, this implies

$$H_1 = a(u) \sum_i (x^i)^2 + B_i(u) x^i + c(u).$$

From (26) it follows that for each i it holds

$$\partial_i A_i = \frac{1}{2} H_1.$$

Integrating this equation, we get

$$A_i = \frac{1}{2} \left(a(u) x^i \sum_{j \neq i} (x^j)^2 + \frac{a(u)}{3} (x^i)^3 + \sum_{j \neq i} B_j(u) x^j x^i + \frac{B_i(u)}{2} (x^i)^2 + c(u) x^i + c_i(x^k, u) \right), \quad \partial_i c_i = 0.$$

Let $i \neq j$. Substituting the obtained A_i to (26), we obtain

$$4a(u) x^i x^j + B_j(u) x^i + B_i(u) x^j + \partial_j c_i + \partial_i c_j = 0.$$

Applying ∂_i , we get

$$4a(u) x^j + B_j(u) + \partial_i^2 c_j = 0.$$

Applying ∂_j , we obtain $a(u) = 0$. We conclude that

$$\partial_i^2 c_j = -B_j(u), \quad \partial_j c_j = 0.$$

This implies

$$c_j = -\frac{B_j(u)}{2} \sum_{k \neq j} (x^k)^2 + d_{jk}(u) x^k + f_j(u), \quad d_{jj}(u) = 0.$$

Using (26) for $i \neq j$, we get

$$d_{ij}(u) = -d_{ji}(u).$$

Thus,

$$H_1 = B_i(u) x^i + c(u), \quad (27)$$

$$A_i = \frac{1}{2} \left(B_j(u) x^j x^i - \frac{B_i(u)}{2} \sum_j (x^j)^2 + c(u) x^i + d_{ik}(u) x^k + f_i(u) \right). \quad (28)$$

Since $\vec{v} \neq 0$, it holds

$$\sum_j B_j^2(u) \neq 0.$$

Consider the coordinate transformation with the inverse one

$$v = \tilde{v}, \quad x^i = \tilde{x}^i + b^i(\tilde{u}), \quad u = \tilde{u}$$

such that $B_i(u)b^i(u) + c(u) = 0$. After that $H_1 = B_i(u)x^i$, i.e. we may assume that $c(u) = 0$. Next, consider the coordinate transformation with the inverse one

$$v = \tilde{v}, \quad x^i = A_j^i(\tilde{u})\tilde{x}^j, \quad u = \tilde{u}, \quad (29)$$

where $A_j^i(u)$ is a family of orthogonal matrices. It is easy to check that

$$\tilde{H}_1 = B_i(u)A_j^i(u)\tilde{x}^j, \quad \tilde{A}_i = \sum_k A_i^k(u)(\partial_u A_l^k(u))\tilde{x}^l + A_i^k(u)A_l^k(u).$$

The obtained metric has the same form and it holds

$$\tilde{B}_i(u) = B_j(u)A_i^j(u), \quad \tilde{d}_{ij}(u) = \sum_k A_i^k(u)\partial_u A_j^k(u) + \frac{1}{2}A_i^r(u)d_{rk}(u)A_j^k(u), \quad \tilde{f}_i(u) = A_i^k(u)f_k(u).$$

Consider the equation $\tilde{d}_{ij}(u) = 0$. Since $\sum_k A_i^k(u)A_j^k(u) = \delta_{ij}$, it can be written in the form

$$\partial_u A_i^k(u) = A_i^j(u)\frac{1}{2}d_{jk}(u).$$

Since $d_{jk}(u)$ is skew-symmetric, $\frac{1}{2}d_{jk}(u)$ is a curve in the Lie algebra $\mathfrak{so}(n)$. Then $A_i^k(u)$ satisfying the above equation is nothing else as the development of the curve $\frac{1}{2}d_{jk}(u)$ in the Lie group $\text{SO}(n)$. Thus, applying such transformation, we may assume that $d_{ij}(u) = 0$. Applying (25), we may assume that $f_i(u) = 0$. Thus,

$$H_1 = B_i(u)x^i, \quad A_i = \frac{1}{2} \left(B_j(u)x^j x^i - \frac{B_i(u)}{2} \sum_j (x^j)^2 \right).$$

Note that

$$F_{ij} = B_i(u)x^j - B_j(u)x^i$$

and (26) holds. The equation $T_{ij} = f\delta_{ij}$ takes the following form:

$$f_0\delta_{ij} = -\frac{1}{2}\partial_i\partial_j H_0 + \frac{1}{4}\sum_k B_k^2(u)x^i x^j + \frac{1}{4}H_1^2\delta_{ij} + \frac{1}{2}\partial_u H_1\delta_{ij}.$$

It can be rewritten in the form

$$\varphi\delta_{ij} = -\frac{1}{2}\partial_i\partial_j H_0 + \frac{1}{4}\sum_k B_k^2(u)x^i x^j,$$

where $\varphi = f_0 - \frac{1}{4}H_1^2 - \frac{1}{2}\partial_u H_1$. Taking the trace, we obtain $\varphi = \frac{1}{n} \left(-\frac{1}{2}\Delta H_0 + \frac{1}{4}\sum_k B_k^2(u)\sum_l (x^l)^2 \right)$, and we get the equation

$$\frac{1}{n} \left(-\frac{1}{2}\Delta H_0 + \frac{1}{4}\sum_k B_k^2(u)\sum_l (x^l)^2 \right) \delta_{ij} = -\frac{1}{2}\partial_i\partial_j H_0 + \frac{1}{4}\sum_k B_k^2(u)x^i x^j.$$

Clearly, the function

$$H_0 = \frac{1}{16}\sum_{k=1}^n B_k^2(u) \left(\sum_{i=1}^n (x^i)^2 \right)^2$$

is a partial solution of this equation. On the other hand,

$$H_0 = a(u) \sum_{i=1}^n (x^i)^2 + C_i(u)x^i + D(u)$$

is the general solution of the corresponding homogeneous system. Thus,

$$H_0 = \frac{1}{16} \sum_{k=1}^n B_k^2(u) \left(\sum_{i=1}^n (x^i)^2 \right)^2 + a(u) \sum_{i=1}^n (x^i)^2 + C_i(u)x^i + D(u).$$

Let us compute the holonomy algebra of the obtained metric. Let $x \in M$ be a point such that $\vec{v}_x \neq 0$. The condition on the curvature tensor shows that

$$R_x(p_x, q_x) = -p_x \wedge \vec{v}_x, \quad R_x(X, Y) = p_x \wedge ((X \wedge Y)\vec{v}_x).$$

This shows that $p_x \wedge E_x \subset \mathfrak{g}$. Next,

$$R_x(\vec{v}_x, q_x) = -g(\vec{v}_x, \vec{v}_x)p_x \wedge q_x - p_x \wedge T_x(\vec{v}_x),$$

which implies $\mathbb{R}p_x \wedge q_x \subset \mathfrak{g}$. Finally,

$$R_x(X, q_x) = -g(\vec{v}_x, X)p_x \wedge q_x + \vec{v}_x \wedge X - p_x \wedge T_x(X).$$

Since the bivectors of the form $\vec{v}_x \wedge X$ generate the Lie algebra $\mathfrak{so}(E_x)$, we conclude that

$$\mathfrak{g} = \mathbb{R}p_x \wedge q_x + \mathfrak{so}(E_x) + p_x \wedge E_x \simeq \mathfrak{sim}(n).$$

Case 2. Suppose now that $s_0 \neq 0$. Since h is independent of v , $\partial_v \lambda = 0$. From (10) it follows that

$$H = \lambda v^2 + H_1 v + H_0, \quad \partial_v H_1 = \partial_v H_0 = 0.$$

From (11) it follows that the components of tensor P do not depend on the coordinate v . This, the equation $P(X) = \vec{v} \wedge X$ and (10) imply that $\partial_i \partial_v^2 H = 0$, i.e. $\partial_i \lambda = 0$, consequently s_0 and λ are functions depending only on u . Note that for $n \geq 3$ this follows also from $\text{Ric}(h) = \frac{s_0}{2} \text{id}$. We conclude that each metric in the family $h(u)$ is of constant sectional curvature.

Case 2.1. Suppose that $s_0 > 0$. Then we may change the coordinates in such a way that

$$h = \frac{1}{-\lambda} \Psi \sum_{k=1}^n (dx^k)^2, \quad \Psi = \frac{4}{(1 + \sum_{k=1}^n (x^k)^2)^2},$$

where $\Psi \sum_{k=1}^n (dx^k)^2$ is the metric on the sphere. Let us introduce the new coordinate \tilde{u} such that $d\tilde{u} = -\lambda(u)du$. The metric g can be written in the form

$$g = \frac{1}{-\lambda} g_0,$$

where g_0 is the metric written in coordinates $v, x^1, \dots, x^n, \tilde{u}$ that has the same form as g and satisfying $\lambda = -1$. We see that it is enough to find conformally flat metrics g such that $\lambda = -1$.

Applying the transformation

$$v \mapsto v - \frac{1}{2} H_1,$$

we get the new metric with $H_1 = 0$ [12].

Let us consider the equation $T_{ij} = f\delta_{ij}$. From (12) it follows that $f = vf_1 + f_0$, $\partial_v f_1 = \partial_v f = 0$. Applying ∂_v to $T_{ij} = f\delta_{ij}$, we get the equation

$$f_1\delta_{ij} = \frac{1}{2}(\nabla_i A_j + \nabla_j A_i).$$

It is enough to consider the equations

$$\nabla_i A_i = \nabla_j A_j, \quad \nabla_i A_j + \nabla_j A_i = 0, \quad i \neq j. \quad (30)$$

The Christoffel symbols of the metric h are the following:

$$\Gamma_{ij}^k = \frac{1}{2\Psi}(\delta_{kj}\partial_i\Psi + \delta_{ki}\partial_j\Psi - \delta_{ij}\partial_k\Psi).$$

Using that, the above equations may be rewritten in the form

$$\partial_i \left(\frac{A_i}{\Psi} \right) = \partial_j \left(\frac{A_j}{\Psi} \right), \quad \partial_i \left(\frac{A_j}{\Psi} \right) + \partial_j \left(\frac{A_i}{\Psi} \right) = 0, \quad i \neq j. \quad (31)$$

We will distinguish the cases $n = 2$ and $n \geq 3$.

Case 2.1.1. Suppose that $n \geq 3$.

Lemma 2 *If $n \geq 3$, then the general solution of the system*

$$\partial_i f_i = \partial_j f_j, \quad \partial_i f_j + \partial_j f_i = 0, \quad i \neq j$$

has the form

$$f_i = x^i B_i x^k - \frac{1}{2} B_i \sum_{k=1}^n (x^k)^2 + d_{ik} x^k + c x^i + c_i,$$

where $B_i, c_i, c, d_{ik} \in \mathbb{R}$, $d_{ki} = -d_{ik}$.

Proof. Let i, j, k be pairwise different, then

$$\partial_i \partial_j f_k = -\partial_i \partial_k f_j = \partial_k \partial_j f_i = -\partial_i \partial_j f_k,$$

i.e. $\partial_i \partial_j f_k = 0$. This shows that $f_k = \sum_{i \neq k} C_{ki}(x^i, x^k)$. Then it is not hard to find these functions. \square

We conclude that

$$A_i = \Psi \left(x^i B_i(u) x^k - \frac{1}{2} B_i(u) \sum_{k=1}^n (x^k)^2 + d_{ik}(u) x^k + c(u) x^i + c_i(u) \right), \quad (32)$$

where $B_i(u), c_i(u), c(u), d_{ik}(u)$, are functions of u , and $d_{ki}(u) = -d_{ik}(u)$.

The system of equations that we have solved is very similar to the equation of the Killing 1-form:

$$\nabla_i A_j + \nabla_j A_i = 0.$$

Let us prove the following lemma.

Lemma 3 Any Killing vector field on the sphere with the metric $\Psi \sum_{k=1}^n (dx^k)^2$ has the following form

$$X = X^i \partial_i, \quad X^i = x^i b_i - \frac{1}{2} b_i \sum_{k=1}^n (x^k)^2 + f_{ik} x^k + \frac{1}{2} b_i,$$

where $b_i, f_{ik} \in \mathbb{R}$, $f_{ki} = -f_{ik}$.

Note that this description corresponds to the fact that the Lie algebra of the Killing vector fields on the n -sphere S^n is isomorphic to $\mathfrak{so}(n+1)$. Recall that S^n can be viewed as the symmetric space $S^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$. The symmetric decomposition of the Lie algebra $\mathfrak{so}(n+1)$ is of the form $\mathfrak{so}(n+1) = \mathfrak{so}(n) + \mathbb{R}^n$. The vector fields defined by the numbers f_{ik} correspond to elements of $\mathfrak{so}(n)$, while the vector fields defined by the numbers b_i correspond to elements of \mathbb{R}^n .

Proof of Lemma 3. Consider the Killing 1-form $A_i = h_{ij} X^j$. In addition to equations (30) it satisfies the equation $\nabla_i A_i = 0$. This equation takes the form

$$\partial_i \left(\frac{A_i}{\Psi} \right) = \frac{1}{2\Psi^2} \sum_{k=1}^n \partial_k A_k.$$

This implies that $A_i = \Psi f_i$, where f_i given by Lemma 2 with $c = 0$ and $c_i = \frac{B_i}{2}$. This proves the Lemma. \square

Let us come back to the 1-form A that is given by (32). As for the case $s_0 = 0$, we may consider the transformation (29) and get $d_{ik}(u) = 0$. Similarly, considering a transformation defined by a Killing vector field depending on u and given by $b_i(u)$, we will get $c(u) = 0$. Thus,

$$A_i = \Psi \left(x^i B_k(u) x^k - \frac{1}{2} B_i(u) \sum_{k=1}^n (x^k)^2 + c_i(u) \right). \quad (33)$$

Now we consider the equation $P(X) = \vec{v} \wedge X$. From (11) and (10) it follows that this equation takes the form

$$\frac{1}{2} \nabla_k F_{ij} = \Psi A_i \delta_{kj} - \Psi A_j \delta_{ki}.$$

Note that

$$F_{ij} = \Psi^{\frac{3}{2}} ((B_i(u) + 2c_i(u))x^j - (B_j(u) + 2c_j(u))x^i).$$

It is easy to check that if $k \neq i$ and $k \neq j$, then $\nabla_k F_{ij} = 0$. Let $k = i$. The equation

$$\frac{1}{2} \nabla_i F_{ij} = -\Psi A_j$$

implies $c_i(u) = \frac{1}{2} B_i(u)$.

We are left with the equation $f_0 \delta_{ij} = T_{ij}$, where

$$T_{ij} = -\frac{1}{2} \nabla_i \nabla_j H_0 + \frac{1}{4} F_{ik} F_{jl} h^{kl} + \frac{1}{2} \partial_u (\nabla_i A_j + \nabla_j A_i) + A_i A_j.$$

Note that if $i \neq j$, then

$$\nabla_i \nabla_j H_0 = \sqrt{\Psi} \partial_i \partial_j \frac{H_0}{\Psi}.$$

We obtain the equation

$$\partial_i \partial_j \frac{H_0}{\Psi} = 2\sqrt{\Psi} B_i(u) B_j(u) - 2\Psi B_k x^k (B_j(u) x^i + B_i(u) x^j) + 2\Psi^{\frac{3}{2}} \left(\sum_{k=1}^n B_k^2(u) + (B_k(u) x^k)^2 \right), \quad i \neq j.$$

The right hand side of this equation can be rewritten as $\partial_i \partial_j (\Psi (\sum_{k=1}^n B_k^2(u) + (B_k(u)x^k)^2))$. We conclude that

$$H_0 = \Psi \left(\sum_{k=1}^n B_k^2(u) + (B_k(u)x^k)^2 \right) + \sum_{k=1}^n f_k(x^k).$$

The condition $T_{ii} = T_{jj}$ implies $\partial_i^2 f_i = \partial_j^2 f_j$. This allows to find the function H_0 .

Case 2.1.2. Suppose now that $n = 2$. The system of equations (31) takes the form

$$\partial_1 \left(\frac{A_1}{\Psi} \right) = \partial_1 \left(\frac{A_2}{\Psi} \right), \quad \partial_1 \left(\frac{A_2}{\Psi} \right) + \partial_2 \left(\frac{A_1}{\Psi} \right) = 0 \quad (34)$$

that implies only that $\frac{A_1}{\Psi}$ and $\frac{A_2}{\Psi}$ are real and imaginary parts of a complex homomorphic function of the variable $x^1 + ix^2$. Consider the equations

$$\nabla_1 F_{12} = -2A_2 \Psi, \quad \nabla_2 F_{21} = -2A_1 \Psi.$$

It is easy to check that

$$\nabla_1 F_{12} = \Psi \partial_1 \frac{F_{12}}{\Psi}, \quad \nabla_2 F_{21} = \Psi \partial_2 \frac{F_{21}}{\Psi}.$$

This implies the equations

$$A_1 = \frac{1}{2} \partial_2 f, \quad A_2 = -\frac{1}{2} \partial_1 f, \quad f = \frac{F_{12}}{\Psi}.$$

Substituting this to the first equation from (34), we get

$$\partial_1 \partial_2 \frac{f}{\Psi} = 0,$$

this implies

$$\frac{f}{\Psi} = f_1(x^1) + f_2(x^2).$$

Using that it is not hard to show that A_1 and A_2 are the same as for $n \geq 3$. The rest of the proof remains the same.

Case 2.2. The case $s_0 < 0$ can be considered in the same way as the case $s_0 > 0$. Now we may assume that $\lambda = 1$ and h is the metric of the Lobachevskian space,

$$h = \Psi \sum_{k=1}^n (dx^k)^2, \quad \Psi = \frac{4}{(1 - \sum_{k=1}^n (x^k)^2)^2}.$$

Lemma (3) takes now the form

Lemma 4 *Any Killing vector field on the Lobachevskian space with the metric $\Psi \sum_{k=1}^n (dx^k)^2$ has the following form*

$$X = X^i \partial_i, \quad X^i = x^i b_k x^k - \frac{1}{2} b_i \sum_{k=1}^n (x^k)^2 + f_{ik} x^k - \frac{1}{2} b_i,$$

where $b_i, f_{ik} \in \mathbb{R}$, $f_{ki} = -f_{ik}$.

This description corresponds to the fact that the Lobachevskian space L^n can be viewed as the symmetric space $L^n = \text{SO}(1, n)/\text{SO}(n)$.

It is not hard to show that the holonomy algebras in the case $s_0 \neq 0$ are $\mathfrak{sim}(n)$.

The theorem is proved. \square

Note that the case $s_0 \neq 0$, $\vec{v} = 0$, $T = 0$ corresponds to the first two metrics from possibility 2) from Theorem 2.

5 The Ricci operator of the obtained metrics

The Ricci operator of the first metric has the form

$$\text{Ric} = na(u)\partial_v \otimes du,$$

in particular, $\text{Ric}^2 = 0$.

In [5], complete conformally flat Lorentzian manifolds (M, g) satisfying the condition

$$[R(X, Y), \text{Ric}] = 0 \quad (35)$$

are studied. It is shown that these manifolds are exhausted by the spaces of constant sectional curvature, by the products of two spaces of constant sectional curvature, and by products of spaces of constant sectional curvature with intervals. The Ricci operator of the first metric from Theorem 1 satisfies (35). More over, such metric is complete, e.g. for $a(u) = 1$, i.e. for the Cahen-Wallach spaces. Thus some metrics in [5] are loosen. In [6], pseudo-Riemannian conformally flat manifolds (M, g) satisfying (35) are studied. It is shown that in addition to the obvious cases, (M, g) may be a complex sphere or a space satisfying $\text{Ric}^2 = 0$. Various examples of conformally flat manifolds with $\text{Ric}^2 = 0$ are constructed in [7].

Let us compute the Ricci operator of the second metric from Theorem 1. Let $B(u) = \sum_{i=1}^n B_i(u)X_i \in \Gamma(E)$. Using (10), (13), (14) and Lemma 1, we get

$$\text{Ric}(p) = 0, \quad \text{Ric}(X) = \frac{n}{2}g(X, B(u))p, \quad \text{Ric}(q) = nfp + \frac{n}{2}B(u)$$

for all $X \in \Gamma(E)$. Using the formulas from Section 4, we get

$$f = -\frac{1}{2} \left(\sum_{k=1}^n B_k^2(u) \right) \left(\sum_{j=1}^n (x^j)^2 \right) + \frac{1}{4}H_1^2 + \frac{1}{2}\partial_u H_1.$$

It holds $\text{Ric}^2 \neq 0$ and $\text{Ric}^3 = 0$. Condition (35) is not satisfied.

The scalar curvature of metric 2) is zero.

For metrics 3) and 4) it holds

$$\text{Ric}(p) = \lambda p, \quad \text{Ric}(X) = ng(X, \vec{v})p - (n-1)\lambda X, \quad \text{Ric}(q) = fp + n\vec{v} + \lambda q,$$

where $\vec{v} = \lambda A_i h^{ij} X_j$ and f is a certain function that can be computed. The scalar curvature equals to $2\lambda + s_0 = -(n-2)(n+1)\lambda$ and it is zero only in dimension four.

6 The case of dimension 4

Applying Theorem 4 to a conformally flat nonflat Lorentzian manifold (M, g) of dimension 4 with the holonomy algebra $\mathfrak{g} \subset \mathfrak{so}(1, 3)$, we obtain that (M, g) must satisfy one of the following conditions:

- 1) $\mathfrak{g} = \mathfrak{so}(1, 3)$;
- 2) $\mathfrak{g} \subset \mathfrak{sim}(2)$, i.e. (M, g) is as in Theorem 1 with $n = 2$;
- 3) $\mathfrak{g} = \mathfrak{so}(1, 1) \oplus \mathfrak{so}(2)$, and (M, g) is locally isometric either to the product of (dS_2, cg_{dS_2}) with (L^2, cg_{L^2}) , or to the product of (AdS_2, cg_{AdS_2}) with (S^2, cg_{S^2}) ;

- 4) $\mathfrak{g} = \mathfrak{so}(1, 2)$, and (M, g) is locally isometric to the product of $(\mathbb{R}, (dt)^2)$ either with (dS_3, cg_{dS_3}) , or with (AdS_3, cg_{AdS_3}) ; or $\mathfrak{g} = \mathfrak{so}(3)$, and (M, g) is locally isometric to the product of $(\mathbb{R}, -(dt)^2)$ either with (S^3, cg_{S^3}) , or with (L^3, cg_{L^3}) .

Here $c > 0$ is a constant, and S^n , L^n , dS_n , AdS_n denote, respectively the sphere, Lobachevskian space, de Sitter space and anti de Sitter space with there standard metrics. The standard Friedmann-Robertson-Walker spacetimes are conformally flat and give examples of holonomy $\mathfrak{so}(1, 3)$ [13].

Possible holonomy algebras of conformally flat 4-dimensional Lorentzian manifolds are classified also in [13]. The first metric from Theorem 1 in dimension 4 is given in [17]. In [13], it is stated that it is an open problem to construct a conformally flat metric with the holonomy algebra $\mathfrak{sim}(2)$ (which is denoted in [13] by R_{14}). An attempt to construct such metric is made in [11], where the following metric is constructed:

$$g = 2dxdt + 4ydx dy - 4zdx dz + \frac{(dy)^2}{2y^2} + \frac{(dz)^2}{2y^2} + 2(x + y^2 - z^2)^2(dt)^2. \quad (36)$$

Using Maple, it is easy to check that the Weyl tensor of this metric is not zero, i.e. this metric is not conformally flat. Next, the authors were looking for a metric written in Walker coordinates. These two arguments suggest that the metric must reed as follows:

$$g = 2dxdt + 4ydt dy - 4zdt dz + \frac{(dy)^2}{2y^2} + \frac{(dz)^2}{2y^2} + 2(x + y^2 - z^2)^2(dt)^2. \quad (37)$$

This metric is conformally flat. Making the transformation

$$x \mapsto x - y^2 + z^2, \quad y \mapsto y, \quad z \mapsto z, \quad t \mapsto t,$$

we obtain

$$g = 2dxdt + 2x^2(dt)^2 + \frac{(dy)^2}{2y^2} + \frac{(dz)^2}{2y^2}. \quad (38)$$

We get that the metric (37) is decomposable and its holonomy algebra coincides with $\mathfrak{so}(1, 1) \oplus \mathfrak{so}(2)$, but not with $\mathfrak{sim}(2)$. Thus in this paper we get metrics with the holonomy algebra $\mathfrak{sim}(2)$ for the first time (even more, recall that we find all such metrics).

The field equations of Nordström's theory of gravitation that appeared before Einstein's theory are the following:

$$W = 0, \quad s = 0,$$

see [16]. The metrics from Theorem 1 in dimension 4 provide examples of solutions of these equations.

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